Math 121 4.2 Logarithmic Functions

Objectives:

Review Math 72

1) Write a logarithm as its equivalent exponential
2) Write an exponential as its equivalent logarithm
3) Recognize and use common logs ($\log x$, base 10) and natural logs ($\ln x$, base e)
4) Use the properties of logs to
   - simplify logs
   - combine log expressions into a single log
   - expand logs into sum, difference and/or multiples of log

5) Solve exponential equations from applications
   - doubling time or tripling time with compound interest
   - drug dosage
   - carbon-14 dating
   - learning time
   - the spread of information

NOTE: There is NO CALCULUS in this section, either!

Write each exponential as its equivalent logarithmic equation

① $e^x = 4$

$\log_e 4 = x$

$\ln 4 = x$

$\boxed{x = \ln 4}$

② $10^x = 4$

$\log_{10} 4 = x$

$\log 4 = x$

$\boxed{x = \log 4}$

③ $1.075^x = 2000$

$\log_{1.075}(2000) = x$

$\boxed{x = \log_{1.075}(2000)}$

Use GC to approximate values of each of the previous answers to the nearest ten-thousandth.

① LN 4

≈ 1.38629

$\approx \boxed{1.3863}$

② LOG 4

$\approx .60205$

$\approx \boxed{.6021}$

③ change of base formula

$$\log_b a = \frac{\log_B a}{\log_B b}$$

choose base B that's on the calculator

$$\log_{1.075}(2000) = \frac{\log(2000)}{\log(1.075)}$$

Be sure to close parentheses

≈ 105.10001

$\boxed{105.1000}$

Newer operating systems can do LOGBASE in the MATH menu.

Write each logarithm as its equivalent exponential equation.

④ $\ln x = 4$

$\log_e x = 4$

$e^4 = x$

$\boxed{x = e^4}$

⑤ $\log x = 4$

$\log_{10} x = 4$

$10^4 = x$

$\boxed{x = 10000}$

⑥ $\log_7 x = 4$

$7^4 = x$

$\boxed{x = 2401}$

Approximate ④ to nearest ten-thousandth.

$x = e^\wedge 4 \approx 54.59815 \approx \boxed{54.5982}$

Logarithmic Properties

Bases b must be $b>0$, $b\neq 0$, $b\neq 1$.

$\log_b 1 = 0$ because $b^0 = 1$

$\log_b b = 1$ because $b^1 = b$

$\log_b b^x = x$ because $b^x = b^x$

$b^{\log_b x} = x$ because $f(x) = b^x$ and $f^{-1}(x) = \log_b x$ un-do each other when composed = "inverse properties"

$f(f^{-1}(x)) = x$ $\qquad b^{\log_b x} = x$

$f^{-1}(f(x)) = x$ $\qquad \log_b b^x = x$

$\log_b a + \log_b c = \log_b ac$

$\log_b a - \log_b c = \log_b \frac{a}{c}$

$k \cdot \log_b a = \log_b a^k$

These properties can be used in either direction: to combine into a single log (for solving a log equation) and to separate into simpler logs (for taking a derivative)

Write as the sum and/or difference of multiples of logs and simplify if possible.

⑦ $\ln\sqrt{x}$

$= \ln x^{1/2}$ rewrite with fraction exponent

$= \boxed{\frac{1}{2}\cdot \ln x}$

⑧ $\ln e^{2x}$

$= \boxed{2x}$ using inverse properties

OR $2x \ln e$

$= 2x \cdot 1$

$= \boxed{2x}$

⑨ $\ln\left(\frac{1}{x^2}\right)$

$= \ln 1 - \ln x^2$

$= 0 - 2\ln x$

$= \boxed{-2\ln x}$

(10) $\ln(x^2-4)$

$= \ln[(x-2)(x+2)]$ factor

$= \boxed{\ln(x-2) + \ln(x+2)}$

write as a single log with coefficient 1.

(10) $\ln(x^5) - 3\ln x$

move coefficient to become exponent of argument

$= \ln(x^5) - \ln(x^3)$

$\uparrow$ divide arguments

$= \ln\left(\frac{x^5}{x^3}\right)$ exponent laws – subtract exp.

$= \ln x^{5-3}$

$= \boxed{\ln x^2}$

$1 \ln x^2$

$\uparrow$ coefficient 1, as indicated.

(11) $\ln\left(\frac{x}{2}\right) + \ln(2)$

$\uparrow$ multiply arguments

$= \ln\left(\frac{x}{2} \cdot 2\right)$

$= \boxed{\ln x}$

Recall from Math 45: Percent Increase and Percent Decrease?

New = Base + %·Base or New = Base − %·Base

Let's call Base = P

rate = % = r (use decimal!)

New = A

$A = P + rP$ $A = P - rP$

factor out P

$A = P(1+r)$ $A = P(1-r)$

Exponential Growth is repeated Percent Increase

$$A = [P(1+r)](1+r)$$
$$= P(1+r)(1+r)$$
$$= P(1+r)^2$$

a second year (or unit of time) multiplies the new amount by $(1+r)$ again.

Over a period of time t:

$$\boxed{A = P(1+r)^t}$$
Exponential growth

$$\boxed{A = P(1-r)^t}$$
Exponential decay

P = starting amount
A = amount after t units of time
r = % change per unit of time

(12) The population of N. Dakota increased 4% annually from 2010-2012. Assuming this trend continues, in how many years will the population

a) grow by 25%
b) double
c) triple

Give an exact answer, then round to nearest hundredth.

Exponential Growth
rate 4% = .04 = r

$$A = P(1+r)^t$$
$$A = P(1+.04)^t$$
$$A = P(1.04)^t$$

grow by 25% means add 25% of P to P to get A

$$.25P + P = A$$
$$.25P + 1P = A$$
$$1.25P = A$$

$$1.25P = P(1.04)^t$$

$$P(1.04)^t = 1.25P$$ swap sides to put exponential on left

$$1.04^t = 1.25$$ divide by P to isolate exponential

$$\log 1.04^t = \log 1.25$$ take logs both sides (or write equivalent exponential)

$$t \cdot \log(1.04) = \log(1.25)$$ log property

$$\boxed{t = \frac{\log 1.25}{\log 1.04} \text{ yrs}}$$

Isolate t by dividing both sides by log 1.04

exact answer.

$t \approx 5.689$

$\boxed{t \approx 5.69 \text{ yrs}}$

b) double means $A = 2P$

$2P = P(1.04)^t$

swap

$P(1.04)^t = 2P$

isolate exponential

$(1.04)^t = 2$

$\log 1.04^t = \log 2$

take logs

$t \cdot \log(1.04) = \log(2)$

log property

$$\boxed{t = \frac{\log(2)}{\log(1.04)} \text{ yrs}}$$

$t = 17.672$

$\boxed{t = 17.67 \text{ yrs}}$

c) triple means $A = 3P$

$P(1.04)^t = 3P$

$1.04^t = 3$

$\log 1.04^t = \log 3$

$t \log 1.04 = \log 3$

$$\boxed{t = \frac{\log 3}{\log 1.04} \text{ yrs}}$$

$t \approx 28.011$

$\boxed{t \approx 28.01 \text{ yrs}}$

(13) Election returns in a city of 1 million people are heard by
$P = 1{,}000{,}000(1-e^{-0.4t})$ people within t hours of first broadcast.
How long will it take for
a) half the city to hear the news
b) everyone to hear the news

} exact, then round to nearest tenth.

a) Half of $1{,}000{,}000 = 500{,}000$

$$500{,}000 = 1{,}000{,}000(1-e^{-0.4t})$$

swap to write exponential on left

$$1{,}000{,}000(1-e^{-0.4t}) = 500{,}000$$

isolate exponential
- divide by 1,000,000
- subtract 1
- div or mult by (-1).

$$1-e^{-0.4t} = \frac{500{,}000}{1{,}000{,}000}$$

$$1-e^{-0.4t} = .5$$

$$-e^{-0.4t} = .5-1$$

$$-e^{-0.4t} = -.5$$

$$e^{-0.4t} = .5$$

write equivalent exponential equation.

$$\ln(.5) = -0.4t$$

$$-0.4t = \ln(.5)$$

isolate variable

$$\boxed{t = \frac{\ln(.5)}{-0.4} \text{ hrs}} \text{ exact}$$

$$t \approx 1.73$$

$$\boxed{t \approx 1.7 \text{ hrs}}$$

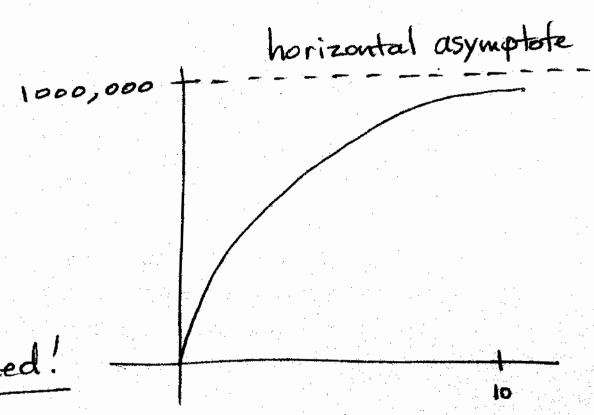
$$1{,}000{,}000 = 1{,}000{,}000(1-e^{-0.4t})$$

$$1 = (1-e^{-0.4t})$$

$$0 = -e^{-0.4t}$$

$$0 = e^{-0.4t}$$

$$\ln(0) = -0.4t$$



ln 0 is undefined!

NEVER get the news to everyone!

(14) The proportion of Carbon-14 still present in a sample after t years is $e^{-0.00012t}$. Estimate the age of the Shroud of Turin from the fact that its linen fibers contain 92.3% of their original carbon-14.

exact, nearest tenth

$$.923 = e^{-0.00012t}$$

$$e^{-0.00012t} = .923$$

$$\ln(.923) = -0.00012t$$

$$\boxed{\frac{\ln(.923)}{-0.00012}\text{ yrs} = t}$$ exact

$$t = 667.71$$

$$\boxed{t \approx 667.7 \text{ yrs}}$$